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On initial conditions, generalized functions and the Laplace transform

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Abstract This note exposes the mathematical setting of initial value problems for causal time-invariant linear systems, given by ordinary differential equations within the framework of generalized functions. We show the structure of the unique solutions for such equations, and apply it to problems with causal or persistent inputs using time-domain methods and generalized Laplace and Fourier transforms. In particular, we correct a widespread inconsistency in the use of the Laplace transform.

Key words Generalized functions – initial value problems – linear systems – Laplace transform – Fourier transform.

1 Introduction

Initial value problems for linear transfer systems in control and systems theory are often given by linear differential equations with constant coefficients and terms that contain derivatives of the input. A generalized functions approach is widely used and mathematically adequate. Recent publications (cf. [1]-[2] and references therein) show that a key issue is the mathematical modelling of the system in question and the treatment of the initial point. The purpose of this contribution is — referring to the work of L. Schwartz (1957) and A.H. Zemanian (1965) — to point out how linear initial value problems with constant coefficients for causal systems can be formulated and solved within the framework of generalized functions in the time-domain and correspondingly in the frequency domain. We discuss system models with causal inputs f, i.e. inputs supported on $[0,\infty[$, and with persistent inputs related to a question raised in [1] for suitable transform methods in that case.

An immediate consequence is the solution with Laplace or Fourier transforms. We observe that for generalized functions f in a series of standard textbooks (see for example [3] - [7]), in numerous course lectures on control and linear systems theory as well as in [2] a modified right-sided Laplace transform, denoted by $\mathcal{L}_-$ and formally defined by

$$\mathcal{L}_-(f) = \int_{0-}^{\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t,$$

is used, so that the generalized derivative f' of a transformable function f with a jump discontinuity at $t_0 = 0$ has the transform

$$\mathcal{L}_-(f')(s) \;=\; s\,\mathcal{L}_-(f)(s) \;-\; f(0-). \qquad (1)$$

Differently from the usual right-sided Laplace transform $\mathcal{L}$ (cf. Section 4), which operates on generalized functions with support in the nonnegative half-line, this $\mathcal{L}_-$ transform and equation (1) are used for functions f with possibly nonzero left-sided limits $f(0-)$. Logically the support of such functions must intersect the negative half-line. As a consequence, the $\mathcal{L}_-$ transform does not fulfill the convolution theorem, a given transform $\mathcal{L}_-(f')$ does not yield a unique primitive f for f' as $\mathcal{L}(f')$ does, and $\mathcal{L}_-$ is not even invertible on generalized functions with a support intersecting the negatives. There are “significant confusions present in many of the standard textbook presentations of this subject” (cf. [2]). In the article of Lundberg et al. [2] with the subtitle “Troubles at the origin” the reader can find an extensive discussion of that confusion in otherwise excellent literature.

We propose to use only the Laplace transform $\mathcal{L}$ as introduced by A.H. Zemanian [8] and L. Schwartz [9], which operates on generalized functions with support in the nonnegative half-line and provides the convolution theorem and well-known correspondence tables. We emphasize that the adequate time-domain model of the initial value problem yields consistent Laplace or Fourier transforms (see Section 4).

For a sufficiently general treatment of usual signals as generalized functions we point the reader to [2] or [10]. The necessary mathematical background is exposed there briefly and elementary enough to be presented in standard courses on engineering mathematics.

2 Causal initial value problems with generalized input in the time-domain

In the sequel we will study the following linear differential equation with constant coefficients

$$P(D)\,y \;=\; Q(D)\,f \qquad (2)$$

for $D = \mathrm{d}/\,\mathrm{d}t$, polynomials $P(\lambda) = \sum_{k=0}^{n} a_k\lambda^k$ ($a_n \neq 0$) and $Q(\lambda) = \sum_{k=0}^{m} b_k\lambda^k$. It is considered to be an equation in the space $\mathcal{D}'$ of generalized functions on $\mathbb{R}$.

When we model by equation (2) a transfer system with given input f and y as output solution to the equation, we have to impose further conditions on the nature of the system and the type of the input, and conditions that determine a unique solution y of (2) as corresponding system output. First, we assume that the system is causal, i.e. an input f to the initially-at-rest system with support in $[t_0, \infty[$ generates a system output y with support in $[t_0, \infty[$. For convenience, an input f is assumed similar to [1] and [2] to be a superposition $f = f_r + f_g$ of a function $f_r \in C^m(\mathbb{R})$ and a generalized function $f_g \in \mathcal{D}'_+$. Here $C^m(\mathbb{R})$ denotes the space of m-times continuously differentiable functions on $\mathbb{R}$, $\mathcal{D}'_+$ the space of generalized functions with support in $[0, \infty[$. We set the initial point $t_0 = 0$, prescribe initial conditions of the form $y^{(k)}(0-) = \lim\limits_{t\to 0, t<0} y^{(k)}(t) = c_k$, $k = 0, \ldots, n-1$, and extend the classical setting of the initial value problem as follows.

Definition 1 *A causal initial value problem for the differential equation* (2) *in* $\mathcal{D}'$ *with* $f = f_r + f_g$, $f_r \in C^m(\mathbb{R})$, $f_g \in \mathcal{D}'_+$, *is to find a generalized function* $y \in \mathcal{D}'$, *which satisfies the following conditions:*

(i) The generalized function y *solves the inhomogeneous equation in* $\mathcal{D}'$.
(ii) For $t < 0$ *the generalized function* y *coincides with the solution* z *of the equation* $P(D)y = Q(D)f_r$, *which has given values* $z^{(k)}(0) = c_k$ *of the* k*-th derivatives* $z^{(k)}$ $(k = 0 \ldots n-1)$.

The following extension of the classical result is probably well-known to the workers in the field. Since we could not trace either a proof nor the statement in the literature, we add a proof that shows the structure of the solution for the initial value problem. As usual δ denotes the Dirac distribution.

Theorem 1 *The solution of the causal initial value problem for equation* (2) *and an input* $f = f_r + f_g$ *with* $f_r \in C^m(\mathbb{R})$, $f_g \in \mathcal{D}'_+$ *is unique and has the form*

$$y \;=\; g * Q(D)f_g \;+\; z. \tag{3}$$

Here g *is the causal fundamental solution of* $P(D)y = \delta$, $g * Q(D)f_g$ *is the convolution of* g *with the generalized function* $Q(D)f_g$, *and* z *is the classical solution of the equation* $P(D)y = Q(D)f_r$, *which satisfies the conditions* $z^{(k)}(0) = c_k$. *The solution* y *then fulfills* $y^{(k)}(0-) = c_k$ $(0 \le k \le n-1)$.

Proof Since the difference of two solutions solves the homogeneous equation for zero initial conditions, a solution is unique and independent of the representation of the superposition $f = f_r + f_g$. The causal fundamental solution g is given by $g = vu$ with the unit step function u and that solution v of $P(D)y = 0$, which satisfies $v(0) = v'(0) = \ldots = v^{(n-2)}(0) = 0$, $v^{(n-1)}(0) = 1/a_n$ (cf. [8] or [9]). Its convolution with $Q(D)f_g$ represents the unique causal solution corresponding to the input f_g for the initially-at-rest system. By linearity and the regularity condition on f_r, the function z adds the unique classical solution of $P(D)y = Q(D)f_r$ establishing the required initial conditions. □

Remark 1 Theorem 1 shows that the unique causal solution as defined in Definition 1 has its support in $[t_0, \infty[$, when the system is initially at rest and the input has its support in $[t_0, \infty[$.

Example 1 For initially-at-rest conditions the equation $y^{(3)} + y' = \delta$ has the unique causal solution $y(t) = (1 - \cos(t))u(t)$ with the unit step function u. It also has a non-causal solution $w(t) = y(t) + \cos(t) - 1$ fulfilling $w(0+) = w'(0+) = w''(0+) = 0$, but $w''(0-) = -1$. Thus, condition (ii) in Definition 1 determines the causal solution as system output with an initial state established by the signal history for $t < 0$.

Example 2 (cf. [1]) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 3\frac{\mathrm{d}f}{\mathrm{d}t} + 5f \tag{4}$$

with $f(t) = 3u(t) - 1$, u the unit step function. Its causal fundamental solution is $g(t) = \mathrm{e}^{-2t}\, u(t)$. For the initial value $y(0-) = -5/2$ the solution on $\mathbb{R}$ according to (3) is

$$y(t) = g * (9\delta + 15u)(t) - \frac{5}{2} = \left(\frac{3}{2}\,\mathrm{e}^{-2t} + \frac{15}{2}\right) u(t) - \frac{5}{2}. \tag{5}$$

3 Problems on half-lines

In application problems we are often interested in predicting the system evolution for $t \geq t_0$, when the system has given initial values at the time t_0. Assuming for simplicity that all input starts at $t_0 = 0$ and considering only the half-line $t \geq 0$, we do not concern ourselves with exactly how the initial conditions are established in a real world system. Mathematically we can assume that the initial values are established by a suitable solution z of the corresponding homogeneous equation. To extend the classical results we ask for a generalized function T with support in $[0, \infty[$, which coincides for m-times continuously differentiable input functions f with the classical solution of the initial value problem on the half-line $t > 0$.

Theorem 2 *(i) For $f \in \mathcal{D}'_+$ the generalized function $T = g * Q(D)f + zu$ is the unique causal solution of the distributional equation*

$$P(D)y \;=\; Q(D)f \;+\; \sum_{k=1}^{n} a_k \left(\sum_{q=0}^{k-1} c_q \delta^{(k-1-q)} \right). \tag{6}$$

Here g denotes the causal fundamental solution of $P(D)y = \delta$, z the classical solution of the homogeneous equation $P(D)y = 0$ satisfying the initial conditions $z^{(k)}(0) = c_k$, $k = 0, \ldots, n-1$, and u is the unit step *function. The solution can also be represented by the convolution of g with the right hand side of equation* (6)

$$T \;=\; g * Q(D)f + \sum_{k=1}^{n} a_k \left(\sum_{q=0}^{k-1} c_q g^{(k-1-q)} \right). \tag{7}$$

(ii) For every m-times continuously differentiable function f with $\mathrm{supp}(f) \subset [0,\infty[$ *the generalized function $T = g * Q(D)f + zu$ is regular and coincides for $t > 0$ with the classical solution y of the causal initial value problem for equation* (2) *with the initial values $y^{(k)}(0-) = c_k$, $k = 0, \ldots, n-1$.*

Proof Equation (6) has a unique solution in $\mathcal{D}'_+$. Substituting $y = T$ into the equation shows the assertion, since for $k = 1, \ldots n$ the following relation holds for the generalized derivatives of zu

$$(zu)^{(k)} \;=\; z^{(k)}u + \sum_{q=0}^{k-1} c_q \delta^{(k-1-q)}. \tag{8}$$

For m-times continuously differentiable functions f with support in $[0,\infty[$ the classical solution of the given initial value problem on $\mathbb{R}$ is the convolution $y = g * Q(D)f + z$. It coincides on the positive half-line with the regular generalized function $T = g * Q(D)f + zu$. □

Theorem 2 shows that equation (6) is the right time domain equation for a causal initial value problem in the framework of generalized functions in $\mathcal{D}'_+$. It extends the classical setting and has already been emphasized by [8] and [11]. This equation contains the initial values c_k explicitly. The influence of these values in an inhomogeneous part of the equation causes the effect of the system initial state to the solution for $t \geq 0$. When we want to analyse the system evolution only for $t \geq 0$, advantages of that equation model are the following:
1) Considering the right-hand side of (6) as input x for the causal initially-at-rest system given in $\mathcal{D}'_+$ by $P(D)T = x$ we have a linear input-output-relation.
2) The initial value problem is now given in the convolution algebra $\mathcal{D}'_+$ of causal distributions. The unilateral Laplace transform $\mathcal{L}$ operates in $\mathcal{D}'_+$ and therefore is a tool for solving the problem. For asymptotically stable systems and tempered inputs the Fourier transform can be used for solving the initial value problem (6) as well.
3) Moreover, completely analogous to equation (6) initial value problems for partial differential equations in a half-space have also been introduced and studied within the framework of generalized functions in [11], Section V.6, and [12], Section 15.
4) Given a persistent input with the assumed regularity properties for f_r (cf.

Section 2) when approaching zero from the left, we can also represent the solution y for $t < 0$ by the parameter transform $t \to -t$ and by the solution of the corresponding reflected initial value problem in $\mathcal{D}'_+$. This allows to find the solution for $t < 0$ also by the right-sided Laplace transform in the following section. For the equation (2) with $f = f_r + f_g$ as before we have

Theorem 3 *For $t < 0$ the solution y of* (2) *with given values $y^{(k)}(0-) = c_k$ ($k = 0, \ldots, n-1$) is the reflection $y(t) = \breve{y}(-t)$, where $\breve{y} \in \mathcal{D}'_+$ is the solution of*

$$P(-D)\breve{y} = (Q(-D)\breve{f}_r)u + \sum_{k=1}^{n}(-1)^k a_k \left(\sum_{q=0}^{k-1}(-1)^q c_q \delta^{(k-1-q)}\right). \tag{9}$$

Proof With v as in the proof of Theorem 1 and $\breve{v}(t) = v(-t)$ its reflection, we observe that $-\breve{v}u$ is the fundamental solution of the reflected equation $P(-D)y = \delta$ in $\mathcal{D}'_+$. Its convolution with the right-hand side of (9) yields the reflection $\breve{y} \in \mathcal{D}'_+$ of the solution y for $t < 0$. Due to the regularity of f_r and v the convolutions $(-\breve{v}u * (Q(-D)\breve{f}_r)u)^{(m)}$ disappear for $m = 0, \ldots, n-1$, when $t \to 0+$. The convolution of $-\breve{v}u$ with the singular term in (9) coincides with $\breve{z}u$, z as in Theorem 2. Therefore the m-th derivative of that convolution tends to $(-1)^m c_m$ for $t \to 0+$. Thus, the reflection of $\breve{y}$ gives the requested initial values $c_m = y^{(m)}(0-)$.

Remark 2 The proof shows that only sufficient regularity properties of f_r near the origin from the left are necessary to obtain the solution in the given form.

Now, we can also solve the initial value problem for suitably transformable inputs by the right-sided Laplace transform or by the Fourier transform in the case of asymptotically stable systems.

4 Transform methods for solving linear causal initial value problems

The right-sided Laplace transform $\mathcal{L}(T)$ of a generalized function $T \in \mathcal{D}'_+$ is defined at $s \in \mathbb{C}$ by applying the functional T to the function $\mathrm{e}_s(t) = \mathrm{e}^{-st}$, usually denoted by

$$\mathcal{L}(T)(s) = T(\mathrm{e}_s) \tag{10}$$

provided that $\mathrm{e}_{x_0} T$ is a tempered distribution for large enough $x_0 \in \mathbb{R}$ and the real part $\Re(s) > x_0$ (cf. [8], [9]).

The most important properties for applications are the invertibility of the Laplace transform and the convolution theorem $\mathcal{L}(T * S) = \mathcal{L}(T) \cdot \mathcal{L}(S)$. This implies immediately the Laplace transform of generalized derivatives T' for transformable $T \in \mathcal{D}'_+$

$$\mathcal{L}(T')(s) = \mathcal{L}(T * \delta')(s) = s\mathcal{L}(T)(s). \tag{11}$$

It does not contain a nonzero pre-initial value $f(0-)$. The essential point, why the Laplace transform works as a tool in solving differential equations in the convolution algebra $\mathcal{D}'_+$, is the property that we can find a unique primitive $T \in \mathcal{D}'_+$ by $\mathcal{L}^{-1}(\mathcal{L}(T')/s)$ for a given transform $\mathcal{L}(T')$. This property as well as the convolution theorem for $\mathcal{L}$ are lost, if nonzero pre-initial values, not intrinsic to the transform, are introduced as in [2] - [7].

Linear initial value problems in $\mathcal{D}'_+$ with constant coefficients are adequately described by equation (6) in Theorem 2. A linear combination of the Dirac distribution δ and its derivatives has the Laplace transform

$$\mathcal{L}\left(\sum_{q=0}^{k-1} c_q \delta^{(k-1-q)}\right) = \sum_{q=0}^{k-1} c_q s^{k-1-q}. \tag{12}$$

Therefore, the Laplace transform of equation (6) yields for transformable generalized functions in $\mathcal{D}'_+$

$$P(s)\mathcal{L}(y)(s) \;=\; Q(s)\mathcal{L}(f)(s) \;+\; \sum_{k=1}^{n} a_k \left(\sum_{q=0}^{k-1} c_q s^{k-1-q}\right). \tag{13}$$

This is the same equation in the image domain of the Laplace transform, which is obtained with nonzero initial values in the differentiation rule (1) for the $\mathcal{L}_-$ transform of generalized derivatives (cf. [2] - [7]). Here the equation (13) is obtained by the usual Laplace transform $\mathcal{L}$ invertible on $\mathcal{D}'_+$. Of course the inverse Laplace transform of (13) gives back the generalized functions equation (6) in the time-domain. In the case of asymptotically stable systems and tempered inputs we can as well use the Fourier transform to solve the initial value problem (6), when we replace the variable s by $i\omega$ ($i^2 = -1$) in (13).

Example 3 (Input with support in the half-line $t \geq 0$) Consider the equation $y'' + 2/\sqrt{LC}y' + 1/(LC)y = U_1 u''$ with initial conditions $y(0-) = U_0$, $y'(0-) = 0$. It describes a simple critically damped RCL circuit ($R^2 = 4L/C$), whose input is the step function $U_1 u(t)$ and the output is the voltage across the inductor. Its causal fundamental solution is $g(t) = \mathrm{e}^{-t/\sqrt{LC}}\, t u(t)$; its causal impulse response is the generalized second derivative $h = g''$.

The solution $y = g * (U_1 u'') + z = h * (U_1 u) + z$ of the initial value problem on $\mathbb{R}$ according to (3) is

$$y(t) = \left(U_1 - \frac{U_1 t}{\sqrt{LC}}\right) \mathrm{e}^{-t/\sqrt{LC}}\, u(t) + \left(U_0 + \frac{U_0 t}{\sqrt{LC}}\right) \mathrm{e}^{-t/\sqrt{LC}}\,. \tag{14}$$

It fulfills $y(0-) = U_0$ and $y(0+) = U_0 + U_1$. For large negative t the solution $y(t)$ is certainly not a physically realistic voltage of the circuit and in general the true system evolution in the past remains unknown. Cutting off the past instead and considering the problem only on the half-line $t \geq 0$ for the given initial conditions, we obtain the solution $T \in \mathcal{D}'_+$ of the generalized equation

$$y'' + \frac{2}{\sqrt{LC}} y' + \frac{1}{LC} y = U_1 u'' + \frac{2U_0}{\sqrt{LC}} \delta + U_0 \delta' \tag{15}$$

according to (6) by

$$T(t) = \left(U_0 + U_1 + \frac{(U_0 - U_1)t}{\sqrt{LC}} \right) \mathrm{e}^{-t/\sqrt{LC}}\, u(t). \tag{16}$$

The Laplace transform of (15) yields $\mathcal{L}(y)(s) = \dfrac{2U_0\sqrt{LC} + LC(U_0 + U_1)s}{(\sqrt{LC}s + 1)^2}$, which has the inverse Laplace transform T in $\mathcal{D}'_+$.

Since the system is asymptotically stable, it has the frequency characteristic $Q(i\omega)/P(i\omega)$, which is a multiplier in the space of tempered distributions and has the causal inverse Fourier transform $h = g''$. Thus, the initial value problem can also be solved by the Fourier transform of equation (15).

Example 4 (Example 2 continued, cf. [1]) For stable systems with persistent inputs as found in [1] and [2], the initial values were reasonably chosen as if the systems were in a steady state due to their "infinitely long lasting history". Therefore the corresponding solution of the equation $P(D)y = Q(D)f$ on $\mathbb{R}$ for a tempered input f can be found by the Fourier transform $\mathcal{F}$ of that equation without any reference to the given initial conditions. Thus, for equation (4) in example 2 the solution y on $\mathbb{R}$ can be represented as

$$y = \mathcal{F}^{-1}(\widehat{y}) \quad \text{with} \quad \widehat{y}(\omega) = \left(\frac{9i\omega + 15}{2i\omega - \omega^2} + \frac{5}{2}\pi\delta \right). \tag{17}$$

Example 5 (Persistent input to an unstable system) We consider the differential equation

$$P(D)y = y^{(4)} + 3y^{(2)} - 6y' + 10y = 2\delta + 2u + 1 \tag{18}$$

with initial conditions $y(0-) = 1$, $y'(0-) = 0$, $y^{(2)}(0-) = 2$, $y^{(3)}(0-) = -1$. According to equation (6) its solution for $t \geq 0$ can be obtained by the Laplace transform of $P(D)y = 2\delta + 3u + (\delta^{(3)} + 5\delta' - 7\delta)$. The inverse of $\mathcal{L}(y)$ yields for $t \geq 0$

$$y(t) = \frac{3}{10} + \frac{1}{130}\,\mathrm{e}^{t}(81\cos(t) + 37\sin(t)) + \frac{1}{65}\,\mathrm{e}^{-t}(5\cos(2t) - 27\sin(2t)). \tag{19}$$

By Theorem 3 the solution for $t < 0$ is the reflection of $\check{y}$, which can be obtained by the right-sided Laplace transform of $P(-D)\check{y} = u + (\delta^{(3)} + 5\delta' + 7\delta)$. The inverse of $\mathcal{L}(\check{y})$ yields for $t < 0$

$$y(t) = \frac{1}{10} + \frac{1}{130}\,\mathrm{e}^{t}(119\cos(t) + 3\sin(t)) - \frac{1}{65}\,\mathrm{e}^{-t}(\cos(2t) + 31\sin(2t)). \tag{20}$$

5 Conclusion

We exposed time-domain equations and solutions for causal linear initial value problems given by differential equations with constant coefficients and generalized inputs. The results can easily be adopted for linear first order systems and for input types having weaker regularity properties than we have used for convenience of the presentation. Transform methods have been shown to be useful for causal and persistent inputs as well. Especially, we hope that our discussion convinces educators of linear systems and control theory and helps in consistently teaching time-domain, Laplace and Fourier transform methods in the framework of generalized functions.

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